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A NOTE ON THE LIMITS OF APPLICATION OF THE REYNOLDS ANALOGY TO GAS FLOW IN TUBES

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Approximate relations are obtained in the form of inequalities allowing determination of the conditions for which the application of a one-dimensional flow model using the Reynolds analogy is not physically valid.

Methods of calculation based on the use of a one-dimensional model and Reynolds analogy are widely employed in engineering practice [1, 2, 3]. It is well known that the Reynolds analogy is applicable only with a number of limit conditions.

A study is made below of the limits of physical validity of methods based on a one-dimensional model and the Reynolds analogy, in the case of variable wall temperature in the flow direction.

All other conditions ensuring Reynolds analogy are assumed to be satisfied.

Let us examine a steady gas flow in a circular cylindrical tube of diameter D, at a section where the flow is fully developed thermally and hydrodynamically.

We shall take as reference velocity, temperature, pressure and density, respectively, the limiting velocity and stagnation temperature, and the pressure and density at the center of the channel entrance; and for the viscosity and eddy viscosity – the viscosity μ_1 on the axis in the entrance determined for the corresponding stagnation temperature. We shall designate by $\widetilde{\mathrm{Re}}$ the Reynolds number based on viscosity μ_1, tube radius D/2, and the limiting velocity at the channel entrance.

We shall base the longitudinal coordinate x on $D\widetilde{\mathrm{Re}}/2$, while the tube radius D/2 will serve as reference scale for the transverse coordinate y. All the values below are dimensionless. The axis Ox coincides with one of the tube generators and is directed downstream, and the axis Oy is directed into the tube. The wall temperature is considered to be constant around the tube perimeter and is a given function of x only, i.e., the symmetrical problem is considered. The gas velocity is subsonic, and the pressure is assumed to be independent of the transverse coordinate.

In the one-dimensional treatment [1, 2, 3] the problem reduces to integration of the set of equations

$$G\frac{du}{dx}+\frac{k-1}{2k}\frac{dp}{dx}+\tau_0=0, \tag{1}$$

$$G\,d\theta/dx=q_0/c_p, \quad p=G(\theta-u^2)/u, \tag{2}$$

where

$$\tau_0=\widetilde{\mathrm{Re}}\,Gu\zeta/4. \tag{3}$$

The resistance coefficient ζ is assumed to be a given function of Re and the temperature factor θ_0/θ, while q_0 is connected with τ_0 by Reynolds analogy $q_0/c_p=\overline{(\theta_0-\theta)\,\tau_0/u}$.

In order to determine the limits of applicability of this method, we shall examine the relation on the tube axis [4]

$$\frac{pu_1}{\theta_1-u_1^2}\frac{du_1}{dx}+\frac{k-1}{2k}\frac{dp}{dx}+f\tau_0=0, \tag{4}$$

$$\frac{d\theta_1}{dx}=F\frac{q_0}{c_p}. \tag{5}$$

Here, from their physical meaning, the coefficients f and $F \geq 0$.

We shall introduce the velocity profile coefficient $\omega = u/u_1$ and assume, as in [1, 2], that it varies only slightly along the flow.

This may be demonstrated on the basis of the following argument.

We approximate the velocity profile by the power law $u_y = u_1 y^n$, where n is a function of Re:

$$u = \frac{1}{G}\int_0^1 \rho u_y^2 (1-y)\,dy = \frac{I_2}{I_1},$$

where

$$I_2 = \int_0^1 \frac{u_y^2(1-y)}{\theta_y - u_y^2}\,dy, \quad I_1 = \int_0^1 \frac{u_y(1-y)}{\theta_y - u_y^2}\,dy.$$

Because of Reynolds analogy

$$(\theta_y - \theta_0)/(\theta_1 - \theta_0) = y^n,$$

and the integrals I_1 and I_2 may be put in the form of the convergent series

$$\frac{1}{2} I_1 = I^{(1,\,-1)} + I^{(3,\,-2)} + I^{(5,\,-3)} + \cdots,$$

$$\frac{1}{2} I_2 = I^{(2,\,-1)} + I^{(4,\,-2)} + I^{(6,\,-3)} + \cdots,$$

where when $0 < \theta_1/\theta_0 < 2$

$$I^{(i,\,j)} = u_1^i \theta_0^j \sum_{k=0}^{\infty} \frac{\binom{j}{k}(\theta_1/\theta_0 - 1)^k}{[n(i+k)+1][n(i+k)+2]},$$

when $\theta_1/\theta_0 > 1$

$$I^{(i,\,j)} = u_1^i \Theta_1^j \sum_{k=0}^{\infty} P_{ik}\binom{j}{k}(\theta_0/\theta_1 - 1)^k,$$

while

$$\binom{j}{0} = 1, \quad \binom{j}{k} = \frac{j(j-1)(j-2)\cdots(j-k+1)}{k!} \quad (k > 0)$$

and

$$P_{ik} = \sum_{l=0}^{k}\left\{(-1)^l \binom{k}{l} \Big/ [n(i+l)+1][n(i+l)+2]\right\}.$$

Results of calculations of ω, based on the relation following from similarity of the profiles

$$(\theta - \theta_0)/(\theta_1 - \theta_0) = \omega, \tag{6}$$

are given in Fig. 1. for n = 0.10 and n = 0.15 (which correspond to Re numbers of the order of 4×10^6 and 4×10^4).

The variation of ω at constant n is negligibly small. Since n depends weakly on Re, we may make the approximation that the value of ω remains constant in each case examined, and its value may be determined for some mean n.

Introducing $\omega = u/u_1$ into (4) and using (3) and (1), we easily obtain

$$f = 1 - \frac{G}{\tau_0} \times \left(\frac{\theta - u^2}{\theta_1\omega^2 - u^2} - 1\right)\frac{du}{dx}. \tag{7}$$

From (1) and (3), taking into account (6), we get

$$\frac{du}{dx} = \frac{\tau_0}{G} \times \frac{\theta_0/\theta - 1 + 2k\lambda^2/(k+1)}{1 - \lambda^2}. \tag{8}$$

Substituting (8) into (7) and making a simple transformation, we have

$$f = 1 - \frac{2k\lambda^2/(k+1) + \theta_0/\theta - 1}{1 - \lambda^2}\,\frac{\varphi}{1 - \varphi - (k-1)\lambda^2/(k+1)}, \tag{9}$$

where

$$\varphi = (1 - \omega)(1 + \omega\,\theta_0/\theta). \tag{10}$$

Since $f \geq 0$, the solution of the problem in the formulation given at the beginning of this paper loses its physical meaning* whenever we have the inequality

$$\varphi\left(\frac{2k}{k+1}\lambda^2 + \frac{\theta_0}{\theta} - 1\right) > (1-\lambda^2)\left(1 - \varphi - \frac{k-1}{k+1}\lambda^2\right). \tag{11}$$

It follows from (10) that $\varphi > 0$. It is therefore immediately clear from (11) that the formulation of the problem is incorrect for any θ_0/θ when the λ are sufficiently close to 1. Since the left side of (11) increases and the right side decreases, as λ increases, the formulation of the problem is incorrect for all $\lambda < 1$, if $\varphi(\theta_0/\theta - 1) > 1 - \varphi$, or $\varphi\theta_0/\theta > 1$.

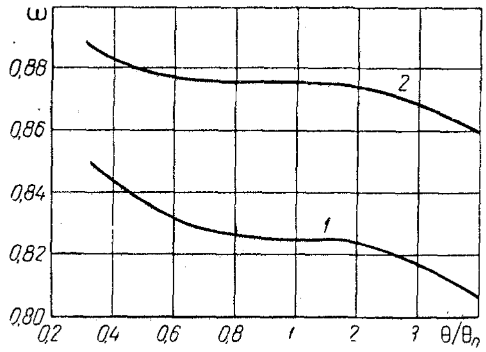

Fig. 1. Dependence of velocity profile coefficient ω on the temperature factor Θ/Θ_0 and Re: 1) Re = 4×10^4; 2) 4×10^6.

Hence, taking (10) into account, we obtain

$$(1-\omega)\,\omega\left(\frac{\theta_0}{\theta}\right)^2 + \frac{\theta_0}{\theta}(1-\omega) - 1 > 0$$

or

$$\frac{\theta_0}{\theta} > \frac{1}{\omega}\left(\sqrt{\frac{1}{4} + \frac{\omega}{1-\omega}} - \frac{1}{2}\right).$$

Thus, for example, when $\omega = 0.85$, we must have $\theta_0/\theta < 2.27$ for the formulation of the problem to be physically valid. It is clear that when the gas is heated, even comparatively small temperature factors lead to an incorrect formulation.

Let us now look at the conditions following from the behavior of coefficient F. From (2) and (6), taking into account that $f \geq 0$, we obtain

$$F = 1 - \frac{Gc_p}{q_0}(1-\omega)\frac{d\theta_0}{dx},$$

whence the condition for incorrectness ($F < 0$) will be

$$\frac{Gc_p}{q_0}(1-\omega)\frac{d\theta_0}{dx} > 1. \tag{12}$$

If the signs of q_0 and θ_0/dx are different, inequality (12) does not hold. Let $d\Theta_0/dx$ and q_0 have the same sign. To obtain easily understandable results we transform (12), using the formulas for τ_0 and q_0/c_p, and obtain

$$\frac{4(1-\omega)}{\widetilde{\mathrm{Re}}\,\zeta(\theta_0-\theta)}\frac{d\theta_0}{dx} > 1. \tag{13}$$

Putting $X = \frac{1}{2}\widetilde{\mathrm{Re}}\,x$, where X is the tube length coordinate referred to its diameter, we have

$$2\frac{1-\omega}{\zeta(\theta_0-\theta)}\frac{d\theta_0}{dX} > 1.$$

For heating, when $\theta < \theta_0$, we must have

$$\frac{1}{\theta_0}\frac{d\theta_0}{dX} > \frac{\zeta}{2}\frac{1-\theta/\theta_0}{1-\omega}. \tag{14}$$

*Henceforth we shall refer to this, for brevity, as the incorrect formulation of the problem.

For cooling

$$\frac{1}{\theta_0}\frac{d\theta_0}{dX} < \frac{\zeta}{2}\frac{1-\theta/\theta_0}{1-\omega}.$$

The inequalities (14) and (15) mean that, for a given value of the temperature factor θ/θ_0, the problem will be incorrectly formulated if the wall temperature increases during heating of the gas and decreases during cooling, faster than the function

$$B(X) = \exp\left(\frac{\zeta}{2}\frac{1-\theta/\theta_0}{1-\omega}X\right).$$

For example, when $\omega = 0.85$ and $\zeta = 0.012$, we have: $B = \exp(0.02X)$ when $\theta/\theta_0 = 0.5$ and $B = \exp(-0.04X)$ when $\theta/\theta_0 = 2$. In the first case the problem will be incorrectly formulated if the wall temperature increases by more than exp (0.02) in a length equal to one diameter, i.e., by 1.02. In the second case, the formulation will be incorrect when the wall temperature decreases in a length of one diameter by more than exp (0.04), i.e., by 1.04. Note that for temperature factors close to 1, the wall temperature gradients are small enough for incorrectness to occur. The more the temperature factor deviates from 1, the less sensitive is the effect of the nonisothermicity of the wall.

To sum up, it may be concluded that the effect of the temperature factor is most appreciable in heating. Note that analysis of (12) leads to the same results.

When the gas is cooled, the correct formulation of the problem breaks down for values of $\lambda = \lambda_K$ fairly close to 1. As the temperature factor approaches 1, the value of λ_K decreases and continues to decrease with further increase of the temperature factor. Finally, at certain values of the temperature factor, as discussed above, the correctness breaks down for any λ_K in the interval [0, 1].

These conclusions are qualitatively quite understandable: it is known that the supply of heat to a gas accelerates subsonic flow and causes an increase (in the absolute value) of the pressure gradient. Conversely, removal of heat has the opposite effect. Therefore, when heat is supplied, the correctness breaks down earlier, and when heat is removed, later than in the adiabatic case.

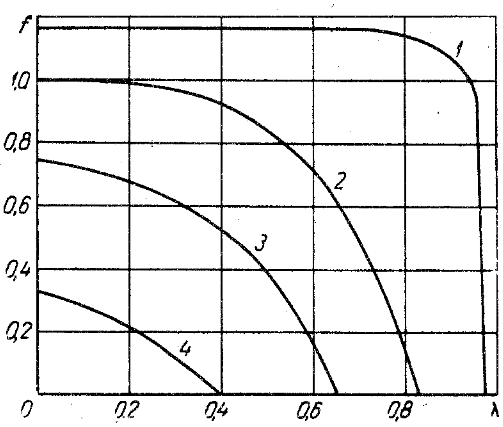

Fig. 2. Dependence of coefficient f on velocity coefficient λ when $\omega = 0.85$: 1) $\theta_0/\theta = 0$; 2) 1; 3) 1.5; 4) 2.

This fact is well illustrated in Fig. 2. Incorrect formulation of the problem corresponds to the region $f < 0$. The curve $\theta_0/\theta = 0$ is the lower boundary of the gas cooling regimes. When $\theta_0/\theta = 0$, the region of λ for which $f < 0$ is limited to a small section near $\lambda = 1$. This region rapidly grows with increase of θ_0/θ, spanning the whole interval [0, 1] when $\theta_0/\theta > 2.27$.

NOTATION

G — mass flow rate of gas; p — pressure; τ — shear stress; θ — stagnation temperature; q — specific heat flux; u — mean mass velocity; $\omega = u/u_1$ — velocity profile coefficient; $\lambda = u/a_*$ — velocity coefficient; f and F — coefficients proportional, respectively, to curvature of velocity profiles and profile of stagnation temperatures on the axis. Subscripts: 0 — wall, 1 — axis of the tube.

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